

Translating solitons from semi-Riemannian foliations

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To the memory of Antonio.

Abstract

In this paper, we introduce a notion of (vertical) translating solitons in a product of a semi-Riemannian manifold (M, g) with the real line. Mainly, we restrict our attention to those which are the graph of a smooth function u . When dealing with submersions, we show a criteria to lift (or project) translating solitons from the base manifold to the total space (or viceversa). In particular, manifolds foliated by codimension 1 orbits of a Lie group action give rise to such solitons, up to solving a first-order ordinary differential equation. This gives us explicit criteria under which the graph of a function is a soliton, and we employ them to construct many examples of solitons, both new and old, in a unified way.

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1 Introduction

Given a smooth manifold M , assume a family of smooth immersions in a semi-Riemannian manifold (\mathbf{M}, \mathbf{g}) , $F_t : M \rightarrow \mathbf{M}$, $t \in [0, \delta)$, $\delta > 0$, with mean curvature vector \vec{H}_t . The initial immersion F_0 is called a solution to the *mean curvature flow* if

$$\frac{d}{dt}F_t = \vec{H}_t. \quad (1)$$

Until now such solutions have been almost exclusively studied in the case where the ambient space is the Euclidean (or the Minkowski) space. For a good list of known examples, see [7]. Probably, the most famous examples are the Grim Reaper curve in \mathbb{R}^2 and the translating paraboloid and translating catenoid, [4]. Also, in [10] there are some examples with complicated topology. But the starting point of this paper is the fact that the translating paraboloid and translating catenoid are rotationally symmetric, namely, invariant by the Lie group $SO(n)$ acting by isometries.

The purpose of this paper is twofold. First, we introduce a definition of translating solitons when the ambient manifold is different from the Euclidean or the Minkowski spaces, in a quite general setting. Second, we are interested in constructing translating solitons having a nice behaviour under the action of Lie groups. We show a technique to construct such examples in different semi-Riemannian manifolds, obviously including the Riemannian case.

Assume that (M, g) is a connected semi-Riemann manifold of dimension $n \geq 2$ and index $0 \leq \alpha \leq n - 1$. Given $\varepsilon = \pm 1$, we construct the semi-Riemannian product $\bar{M} = M \times \mathbb{R}$ with metric $\langle \cdot, \cdot \rangle = g + \varepsilon dt^2$. The vector field $\partial_t \in \mathfrak{X}(\bar{M})$ is obviously Killing and unit, spacelike when $\varepsilon = +1$ and timelike when $\varepsilon = -1$. Now, let $F : \Gamma \rightarrow \bar{M}$ be a submanifold with mean curvature vector \vec{H} . Denote by ∂_t^\perp the normal component of ∂_t along F .

Definition 1. *With the previous notation, we will call F a (vertical) translating soliton of the mean curvature flow, or simply, a translating soliton, if $\vec{H} = \partial_t^\perp$.*

To justify this definition we remark that if $F(x) = (F_1(x), F_2(x))$ for any $x \in \Gamma$, we can define the map

$$F : \Gamma \times \mathbb{R} \rightarrow \bar{M}, \quad F(x, t) = (F_1(x), F_2(x) + t).$$

Hence each $F(-, t)$ is a submanifold with associated mean curvature vector \vec{H}_t and therefore, up to tangential diffeomorphisms, provides a forever solution to (1) invariant by the Killing vector field ∂_t . This clearly generalizes the classical definition of translating solitons in \mathbb{R}^{n+1} . Notice that due to the rich group of isometries of \mathbb{R}^{n+1} , considering any constant unit vector field v such that $\vec{H} = v^\perp$ is equivalent to considering $v = \partial_t$.

In this paper, we will focus on *graphical* translating solitons. Namely, given $u \in C^2(M)$, we construct its graph map $F : M \rightarrow M \times \mathbb{R}$, $F(x) = (x, u(x))$. If ν is the upward normal vector along F with $\varepsilon' = \text{sign}(\langle \nu, \nu \rangle) = \pm 1$, we characterize in Proposition 1 the graphical translating solitons in $(\bar{M}, \langle \cdot, \cdot \rangle)$ as those satisfying the following PDE:

$$\text{div} \left(\frac{\nabla u}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}}. \quad (2)$$

This equation is in general difficult to solve. The geometric way to reduce the number of variables is to consider on the manifold M a foliation by orbits of a Lie group action. But we can often understand the actions by Lie groups as particular cases of submersions. Thus, we first consider the general situation where M submerses to a base manifold B and prove the following result:

Let $\pi : (M, g_M) \rightarrow (B, g_B)$ be a harmonic semi-Riemannian submersion. Given $u \in C^2(B)$, let $F : B \rightarrow B \times \mathbb{R}$, $F(x) = (x, u(x))$ be its graph map, and $\tilde{F} : M \rightarrow M \times \mathbb{R}$ be the map $\tilde{F}(x) = (x, \pi \circ u(x))$. Then \tilde{F} is a graphical translating soliton if and only if F is a graphical translating soliton.

In fact, we obtain a more general result in Theorem 1. Indeed, if we take into account the mean curvature of the fibers in the direction of the normal to the map \tilde{F} , we show that the existence of the translating soliton \tilde{F} in M is equivalent to the existence of what we call an *H-perturbed translating soliton* F in B .

Moreover, an important particular case arise when $\pi : M \rightarrow B = I$, where $B = I$ is an open interval with metric $\tilde{\varepsilon} ds^2$ ($\tilde{\varepsilon} = \pm 1$). Clearly, the fibers of the submersion will be hypersurfaces, and we assume that each fiber $\pi^{-1}(s)$, $s \in I$, has constant mean curvature $h(s)$. Since any function u defined on M which is constant along the fibers, projects to another function $f : I \rightarrow \mathbb{R}$ such that $u = f \circ \pi$, we show in Theorem 2 that equation (2) reduces to the following ODE,

$$f''(s) = (\tilde{\varepsilon} + \varepsilon f'(s)^2)(1 - f'(s)h(s)). \quad (3)$$

Next, we let a Lie group Σ act on the manifold in a nice way. In [9], it is shown that when Σ is compact, and there is at least a good orbit with codimension 1, then, the space of orbits is either

\mathbb{S}^1 or an interval. We will work in the general setting where Σ acts by isometries and the projection to the space of orbits $\pi : M \rightarrow M/\Sigma$ is a well-defined smooth map, where M/Σ is diffeomorphic to an open interval. We show that if the gradient of the projection is nowhere lightlike, then up to composing with another function, π can be assumed to be a semi-Riemannian submersion with constant mean curvature fibers. We state here a more geometric and milder version of Theorem 3, which is a consequence of Theorem 2.

Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : (M, g_M) \rightarrow (I, \tilde{\varepsilon} ds^2)$ be a semi-Riemannian submersion such that the fibers of π are orbits of the action, with function $h : I \rightarrow \mathbb{R}$ representing the mean curvature of the fibers. Given $u \in C^2(M)$, consider its graph map $F : M \rightarrow M \times \mathbb{R}$, $F(x) = (x, u(x))$ for any $x \in M$. Then, F is a Σ -invariant translating soliton if, and only if, there exists a solution $f \in C^2(I, \mathbb{R})$ to (3) such that $u = f \circ \pi$.

This leads to the existence of translating solitons which are the union of two graphical ones, as stated in Corollary 3. We recall that the *translating catenoid* in \mathbb{R}^3 is obtained in this way in [4], and it is one of the main classical examples. Thus, we will be able to obtain translating solitons with two ends in some manifolds. Second, we show in Corollary 4 that, under certain assumptions on the signature and for certain values of the initial value problems, solutions exist on the whole manifold M . In Corollaries 5, 6 and 7, we exhibit some criteria to obtain finite time blow ups for the solutions to (3).

Last Section is devoted to obtaining specific examples. All of them share a common technique: We start with a manifold M and a Lie group of isometries Σ such that the space of orbits M/Σ is an open interval. The associated ODE (3) deeply relies on the auxiliary function $h : M/\Sigma \rightarrow \mathbb{R}$, which depends on the case. When the ODE becomes a problem with a singularity, we show the existence of local solutions by using dynamical systems, and we lift them up to the original manifold M to obtain translating solitons. In some cases, we are able to prove either global existence of the solutions or finite time blow ups. Inspired by the Grim Reaper Cylinder in \mathbb{R}^n , we construct many others by using product manifolds. Needless to say, we recover the already known rotationally symmetric translating solitons in the Euclidean and Minkowski Spaces, but we obtain among others a new *translating catenoid* in Minkowski Space, rotationally symmetric translating solitons in the De Sitter Space and the Hyperbolic Space. By a gluing technique from [1], we construct a C^∞ , boost invariant, translating soliton in Minkowski Space by gluing (up to) 4 pieces. The goal is not to provide an exhaustive list, but rather to illustrate our technique, and make it useful for future works.

2 Setup

The following proposition is well-known in the Euclidean setting (see for example [6]). The proof is similar in our context, but we include it for completeness and discuss the necessary modifications.

Proposition 1. *Let (M, g) be a semi-Riemannian manifold, $u : M \rightarrow \mathbb{R}$ a C^2 function, and let $F : M \rightarrow M \times \mathbb{R} =: \bar{M}$, $F(x) = (x, u(x))$ be its graph map. Given $\varepsilon = \pm 1$, assume that $F : (M, \gamma = F^* \langle, \rangle) \rightarrow (\bar{M}, \langle, \rangle = g + \varepsilon dt^2)$ is a semi-Riemannian hypersurface with unit upward normal ν such that $\langle \nu, \nu \rangle = \varepsilon' = \pm 1$. Then, F is a (vertical) translating soliton if, and only if, function u satisfies (2).*

Proof. Note that, under the usual identifications, for each $X \in TM$, we have

$$dF(X) = (X, du(X)) = (X, 0) + du(X)\partial_t = (X, g(\nabla u, X)) = (X, 0) + g(\nabla u, X)\partial_t,$$

where ∇u is the g -gradient of u . We consider the metric $\gamma = F^* \langle, \rangle$ on M , but we have to assume that (M, γ) is a semi-Riemannian submanifold. In particular, the following function is constant, $\varepsilon' = \text{sign}(\varepsilon + |\nabla u|_g^2) = \pm 1$. Therefore, the upward normal vector field is

$$\nu = \frac{1}{W}(-\varepsilon(\nabla u, 0) + \partial_t), \quad W = +\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}. \quad (4)$$

Needless to say, $\langle \nu, \nu \rangle = \varepsilon'$. Then, we consider a local g -orthonormal frame $B = (e_1, \dots, e_n)$ such that $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, for any $i, j = 1, \dots, n$. We denote $u_i = du(e_i)$, $i = 1, \dots, n$. For this frame, we compute the induced metric $\gamma = F^* \langle, \rangle$, so that the coefficients of the Gram matrix are

$$\gamma_{ij} = \gamma(e_i, e_j) = \varepsilon_i \delta_{ij} + \varepsilon u_i u_j.$$

Then, the inverse matrix is

$$\gamma^{ij} = \varepsilon_i \delta_{ij} - \frac{\varepsilon'}{W^2} \varepsilon_i \varepsilon_j u_i u_j.$$

Now, if II is the second fundamental form of F , then the mean curvature vector of F is

$$\vec{H} = \text{tr}_\gamma(II) = \varepsilon' \langle \vec{H}, \nu \rangle \nu = \varepsilon' \langle \partial_t^\perp, \nu \rangle \nu = \varepsilon' \langle \partial_t, \nu \rangle \nu = \frac{\varepsilon \varepsilon'}{W} \nu.$$

But in our setting, what really matters is $\langle \nu, \vec{H} \rangle = \frac{\varepsilon}{W}$. On the other hand, let $\bar{\nabla}$ the Levi-Civita connection of $(\bar{M}, \langle, \rangle)$. We recall O'Neill's book [8], and its equations for the Levi-Civita connection of a (warped) product. Thus,

$$\begin{aligned} \langle \nu, \vec{H} \rangle &= \langle \nu, \text{tr}_\gamma(II) \rangle = \sum_{i,j} \gamma^{ij} \langle \nu, \bar{\nabla}_{dF(e_i)} dF(e_j) \rangle = \sum_{i,j} \gamma^{ij} \frac{\varepsilon}{W} du_j(e_i) \\ &= \frac{\varepsilon}{W} \text{div}(\nabla u) - \varepsilon \varepsilon' \sum_{i,j} \frac{\varepsilon_i \varepsilon_j u_i u_j}{W^3} e_i(e_j(u)). \end{aligned}$$

Now, we compute

$$\left\langle \nabla \left(\frac{1}{W} \right), \nabla u \right\rangle = \frac{-1}{W^2} \sum_i \varepsilon_i u_i e_i \left(\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)} \right) = \frac{-\varepsilon'}{W^3} \sum_{i,j} \varepsilon_i \varepsilon_j u_i u_j e_i(e_j(u)).$$

All together,

$$\langle \nu, \vec{H} \rangle = \frac{\varepsilon}{W} \text{div}(\nabla u) + \varepsilon \left\langle \nabla \left(\frac{1}{W} \right), \nabla u \right\rangle = \varepsilon \text{div} \left(\frac{\nabla u}{W} \right).$$

□

Remark 1. If we call $H = \text{tr}(A)/n$ the mean curvature function of the hypersurface, A being the shape operator, then $\langle \nu, \vec{H} \rangle = nH\varepsilon'$. In particular,

$$\text{div} \left(\frac{\nabla u}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} = \varepsilon' nH.$$

3 Submersions

We first recall shortly some facts about submersions. For more details we refer to [5] and the classical O'Neill's book [8]. If $\pi : (M, g_M) \rightarrow (B, g_B)$ is a semi-Riemannian submersion, we can consider the following decomposition of the tangent bundle TM

$$TM = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{V} := \ker d\pi$ is the vertical distribution consisting of vectors tangent to the fibers and \mathcal{H} is the horizontal complement with respect to g_M . Note that \mathcal{V} is integrable, but it is usually not true for the horizontal distribution. Every vector field X in TM can then be uniquely written as $X = \mathcal{H}X + \mathcal{V}X$ where $\mathcal{H}X$ (resp. $\mathcal{V}X$) is the horizontal (resp. vertical) component. For completeness and clearness sake, we add the following

Definition 2. A vector field $X \in \Gamma(TM)$ is called *horizontal*, if for all $x \in M$, $X_x \in \mathcal{H}_x$ and *vertical*, if for all $x \in M$, $X_x \in \mathcal{V}_x$. It is called *projectable*, if there exists a vector field $\check{X} \in \Gamma(TB)$ such that for all $x \in M$, $d\pi(X_x) = \check{X}_{\pi(x)}$, X and \check{X} are called π -related. It is called *basic*, if it is projectable and horizontal.

We add the following useful

- Remark 2.** 1) Obviously, from the definition of \mathcal{V} , a vector field $X \in \Gamma(TM)$ is vertical, if and only if it is π -related to the null section of TB .
- 2) For every vector field \check{X} in TB there exists a unique horizontal vector field in $\Gamma(TM)$ which is π -related to \check{X} . It is called the horizontal lift of \check{X} . We will denote it by X^h . Moreover we will denote by $\mathcal{H}(X)$ the horizontal component of a vector field X of TM .
- 3) If X, Y are basic vector fields on M π -related to \check{X}, \check{Y} , then $g_M(X, Y) = \pi^*g_B(\check{X}, \check{Y})$. Moreover $\mathcal{H}\nabla_X^M Y$ is the basic vector field corresponding to $\pi^*\nabla_{\check{X}}^B \check{Y}$, where ∇^M (respectively ∇^B) are the Levi-Civita connection on M (respectively B).

Finally let W, Z be vector fields in TM . We recall the definition of the two O'Neill fundamental tensors.

$$\begin{aligned} T_W Z &:= \mathcal{H}\nabla_{\mathcal{V}W}^M(\mathcal{V}Z) + \mathcal{V}\nabla_{\mathcal{V}W}^M(\mathcal{H}Z), \\ A_W Z &:= \mathcal{V}\nabla_{\mathcal{H}W}^M(\mathcal{H}Z) + \mathcal{H}\nabla_{\mathcal{H}W}^M(\mathcal{V}Z) \end{aligned} \quad (5)$$

Note that if W and Z are vertical, T is the second fundamental form of the fibers. Hence the fibers of the submersion are totally geodesic if and only if $T \equiv 0$. We recall that the submersion is harmonic, if and only if the mean curvature of the fiber $h = 0$. Further, let X, Y be horizontal vectors, and U, V be vertical vectors we recall the following useful formulas

$$\nabla_U^M V = T_U V + \hat{\nabla}_U V, \quad \nabla_U^M X = \mathcal{H}\nabla_U^M X + T_U X \quad (6)$$

$$\nabla_X^M U = A_X U + \mathcal{V}\nabla_X^M U, \quad \nabla_X^M Y = \mathcal{H}\nabla_X^M Y + A_X Y \quad (7)$$

where $\hat{\nabla}$ is the connection on the fibers.

Theorem 1. Let $\pi : (M, g_M) \rightarrow (B, g_B)$ be a semi-Riemannian submersion with constant mean curvature fibers. Given $u \in C^2(B)$, let $F : B \rightarrow B \times \mathbb{R}$, $F(x) = (x, u(x))$ be its graph map, and $\tilde{F} : M \rightarrow M \times \mathbb{R}$ be the map $\tilde{F}(x) = (x, u \circ \pi(x))$. Then \tilde{F} is a graphical translating soliton if and only if F satisfies the equation $\tilde{H}_F = \partial_t^\perp + H_{\text{fib}}^{\nu_{\tilde{F}}} \nu_F$.

Proof. Consider the submersion $\pi : (M, g_M) \rightarrow (B, g_B)$. $\tilde{F}(x) = (x, v(x))$ for any $x \in M$, where $v : M \rightarrow \mathbb{R}$ is the height function. Similarly, $F(p) = (p, u(p))$ for $p \in B$, $u : B \rightarrow \mathbb{R}$, and by assumption $u \circ \pi = v$.

Now locally for every $x_0 \in M$, we take an open neighborhood U of x_0 and a collection of projectable vector fields $\{e_i\}_{i=1}^n$ such that $\{e_i(x)\}_{i=1}^n$ is a basis for $T_x M$, for all $x \in U$. Note that we can assume that e_1, \dots, e_k are vertical and e_{k+1}, \dots, e_n are horizontal w.r.t. π . Then by Remark 2

$$g_M((\nabla^B u)^h, e_i) = \pi^* g_B(\nabla^B u, d\pi(e_i)) = \pi^* du(d\pi(e_i)) = dv(X_i) = g_M(\nabla^M v, e_i),$$

Hence $(\nabla^B u)^h = \nabla^M v$ and, since the submersion is semi-Riemannian, $|\nabla^M v|^2 = |\nabla^B u|^2$.

Next, $\pi \times 1 : \bar{M} = (M \times \mathbb{R}, g_M + \varepsilon dt^2) \rightarrow \bar{B} = (B \times \mathbb{R}, g_B + \varepsilon dt^2)$ is another semi-Riemannian submersion, satisfying that the horizontal lift of ∂_t is ∂_t . Since $|\nabla^M v|^2 = |\nabla^B u|^2$, we have $W^2 = \varepsilon'(\varepsilon + |\nabla^M v|^2) = \varepsilon'(\varepsilon + |\nabla^B u|^2)$. Notice that obviously by construction $\nu_{\tilde{F}}$ is horizontal. Moreover for any $(p, t) \in B \times \mathbb{R}$, $(\pi \times 1)^{-1}(p, t) = \pi^{-1}(p) \times \{t\}$. By equation (4), the horizontal lift w.r.t. $\pi \times 1$ of the normal ν_F in $B \times \mathbb{R}$ is exactly the normal $\nu_{\tilde{F}}$ in $M \times \mathbb{R}$.

By extending the basis $\{e_i\}_{i=1}^n$ of TM , we can now construct the local orthonormal frame $(e_1, \dots, e_n, e_{n+1} = \partial_t)$ of $T\bar{M}$, where $\varepsilon_i = g_{\bar{M}}(e_i, e_i) = \pm 1$, depending on the signature of M . Note that $\{v_i := d(\pi \times 1)e_i | k+1 \leq i \leq n+1\}$ is a local orthonormal frame of $T\bar{B}$ with same signs. Now for the vertical vector fields e_i , $1 \leq i \leq k$, and the horizontal vector field $\nu_{\tilde{F}}$ we get using formulas (6)

$$g_{\bar{M}}(\nabla_{e_i}^{\bar{M}} \nu_{\tilde{F}}, e_i) = g_{\bar{M}}(\mathcal{H} \nabla_{e_i}^{\bar{M}} \nu_{\tilde{F}}, e_i) + g_{\bar{M}}(T_{e_i} \nu_{\tilde{F}}, e_i),$$

Hence $\nabla_{e_i}^{\bar{M}} \nu_{\tilde{F}}$ is horizontal if and only if the second term of the right hand side vanishes. Since the vector fields e_i are vertical, they are by definition tangent to the leaves of the submersion, and the horizontal vector field $\nu_{\tilde{F}}$ is normal to the leaves. Hence $\sum_i^k g_{\bar{M}}(T_{e_i} \nu_{\tilde{F}}, e_i)$ is exactly the mean curvature $H_{\text{fib}}^{\nu_{\tilde{F}}}$ of the fibers in the direction of $\nu_{\tilde{F}}$.

Denoting hence respectively by $II_{\tilde{F}}$ and II_F the second fundamental forms of \tilde{F} and F with associated mean curvature vectors $\vec{H}_{\tilde{F}}$ and \vec{H}_F we use Remark 2 to compute

$$\begin{aligned} \vec{H}_{\tilde{F}} &= \sum_i^n \varepsilon_i II_{\tilde{F}}(e_i, e_i) = \sum_i^n \varepsilon_i g_{\bar{M}}(\nabla_{e_i}^{\bar{M}} \nu_{\tilde{F}}, e_i) \nu_{\tilde{F}} = \sum_{i \geq k+1}^n \varepsilon_i g_{\bar{M}}(\nabla_{e_i}^{\bar{M}} \nu_{\tilde{F}}, e_i) \nu_{\tilde{F}} + H_{\text{fib}}^{\nu_{\tilde{F}}} \nu_{\tilde{F}} \\ &= \sum_{i \geq k+1} \varepsilon_i \pi^* g_{\bar{B}}(\pi^* \nabla_{v_i}^{\bar{B}} \nu_F, v_i) \nu_F^h + H_{\text{fib}}^{\nu_{\tilde{F}}} \nu_{\tilde{F}} = \left(\sum_{i \geq k+1} \varepsilon_i g_{\bar{B}}(\nabla_{v_i}^{\bar{B}} \nu_F, v_i) \nu_F \right)^h + H_{\text{fib}}^{\nu_{\tilde{F}}} \nu_{\tilde{F}} \\ &= \vec{H}_F^h + H_{\text{fib}}^{\nu_{\tilde{F}}} \nu_{\tilde{F}}. \end{aligned}$$

Projecting and using the translating soliton equation then directly shows our result. Note that the projection of the mean curvature of the fibers to the base only makes sense if it is constant. \square

Remark 3. We notice that this result especially holds in the case where the submersion has totally geodesic fiber or is harmonic. In fact, the horizontal lift of \vec{H}_F is $\vec{H}_{\tilde{F}}$ if and only if the fibers are minimal with respect to the normal vector $\nu_{\tilde{F}}$, i.e. $H_{\text{fib}}^{\nu_{\tilde{F}}} = 0$. We point out that only the minimality in the direction of $\nu_{\tilde{F}}$ is needed. If for every point $x \in M$ there is a basis of tangent vectors for $T_{(x,0)}(B \times \mathbb{R})$ and graphical solitons with these as normal vectors then the harmonicity of the submersion π is equivalent to the condition that, for every u , F is a graphical soliton if and only if \tilde{F} is so.

From Theorem 1 we get immediately the following

Corollary 1. *Let $\pi : (M, g_M) \rightarrow (B, g_B)$ be a harmonic semi-Riemannian submersion. Let $u \in C^2(B)$, $F : B \rightarrow B \times \mathbb{R}$, $F(x) = (x, u(x))$ be its graph map, and $\tilde{F} : M \rightarrow M \times \mathbb{R}$ be the map $\tilde{F}(x) = (x, \pi \circ u(x))$. Then \tilde{F} is a graphical translating soliton if and only if F is a graphical translating soliton.*

In view of the preceding theorem, we will call F an H -perturbed soliton. Obviously, by equation (2), F is an H -perturbed soliton if and only if u satisfies

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} - H_{\text{fib}}^{\nu_{\tilde{F}}} \quad (8)$$

We are now going to specialize to the case where the base B is one-dimensional. We will show that in that case equation (8) can be reduced to a particular ODE.

We consider a semi-Riemannian submersion $\pi : (M, g_M) \rightarrow (I, \tilde{\varepsilon} ds^2)$, where I is an open interval. Needless to say, for each $s \in I$, the fiber $\pi^{-1}(s)$ is a hypersurface in M . We assume that each fiber has constant mean curvature (CMC), so that for each $s \in I$, we can call $h(s)$ the value of the mean curvature of $\pi^{-1}(s)$. Along the paper, we will say that h represents the mean curvature of the fibers.

Theorem 2. *Let I be an open interval and $\pi : (M, g_M) \rightarrow (I, \tilde{\varepsilon} ds^2)$ be a semi-Riemannian submersion with CMC fibers, and function h representing the mean curvature of the fibers. Given $u \in C^2(M)$ which is constant along the fibers of π , let $\tilde{F} : M \rightarrow M \times \mathbb{R}$, $\tilde{F}(x) = (x, u(x))$, be its graph map, and $f : I \rightarrow \mathbb{R}$ be the map such that $u = f \circ \pi$. Then \tilde{F} is a translating soliton if and only if function f is a solution to (3).*

Proof. By Theorem 1, and since the base is one-dimensional, F is a translating soliton if and only if

$$\tilde{\varepsilon} \left(\frac{f'}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} \right)' = \frac{1}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} - H_{\text{fib}}^{\nu_{\tilde{F}}} \quad (9)$$

Since π is a semi-Riemannian submersion, $|\nabla \pi|^2 = \tilde{\varepsilon}$, where $\tilde{\varepsilon} = \pm 1$ matches the signature of the base I . Then we have $(f \circ \pi)' = (f' \circ \pi) \nabla \pi$. After computing out the term on the left-hand side of equation (9), we get therefore

$$\tilde{\varepsilon} \frac{f''}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} \left(1 - \frac{\tilde{\varepsilon} f'^2}{\varepsilon + \tilde{\varepsilon} f'^2} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} - H_{\text{fib}}^{\nu_{\tilde{F}}}$$

Let us now point out the following fact about mean curvature. Let H^N be the mean curvature of a submanifold (Z, g_Z) with respect to a unit normal vector $N = \sum_i a_i \xi_i$, where $\{\xi_i\}$ are unit normal vectors to the submanifold and a_i are functions on the submanifold. Then if $\{e_i\}$ is an orthonormal basis of the tangent bundle to the submanifold we have by definition

$$H^N = \sum_i g_Z(\nabla_{e_i}(\sum_j a_j \xi_j), e_i) = \sum_i \sum_j a_j g_Z(\nabla_{e_i} \xi_j, e_i) + g_Z(\partial_{e_i}(a) \xi_i, e_i) = \sum_i a_i H^{\xi_i}$$

We now compute the mean curvature $H_{\text{fib}}^{\nu_{\tilde{F}}}$ of the leaves $\pi^{-1}(x)$ in the direction of the normal vector in $M \times \mathbb{R}$. We have that

$$\nu_{\tilde{F}} = \frac{1}{\sqrt{\varepsilon'((\nabla(f \circ \pi))^2 + \varepsilon)}} (\nabla(f \circ \pi), -1) = \frac{1}{\sqrt{\varepsilon'((f' \circ \pi)^2 |\nabla \pi|^2 + \varepsilon)}} ((f' \circ \pi) \nabla \pi - \partial_t)$$

Using the preceding computations, and by the fact that the mean curvature vanishes in the direction of ∂_t , we get consequently

$$H_{\text{fib}}^{\nu_{\tilde{F}}} = \frac{(f' \circ \pi)}{\sqrt{(f' \circ \pi)^2 \tilde{\varepsilon} + \varepsilon}} H_{\text{fib}}^{\nabla \pi}.$$

Now the mean curvature of the fibers inside of M is the mean curvature with respect to the unit normal vector $\nabla \pi$. Since the leaves have constant mean curvature, there exists a function $h : I \rightarrow \mathbb{R}$, such that $H_{\text{fib}}^{\nabla \pi} = h \circ \pi$. Hence equation (9) becomes after computing out the term on its left-hand side

$$\tilde{\varepsilon} \frac{f''}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} \left(1 - \frac{\tilde{\varepsilon} f'^2}{\varepsilon + \tilde{\varepsilon} f'^2} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} - \frac{f'}{\sqrt{\varepsilon'(\varepsilon + \tilde{\varepsilon} f'^2)}} h.$$

and after simplifying

$$f'' = (\tilde{\varepsilon} + \varepsilon f'^2)(1 - f' h),$$

which proves the result. \square

4 Lie Groups

A particular example where the condition of the previous theorem are satisfied is the case of a manifold with a Lie group acting by isometries whose orbits give a foliation by codimension one submanifolds. Note that we have CMC fibers, so we can construct the map h representing the mean curvature of the fibers.

Proposition 2. *Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : M \rightarrow I$ be a submersion such that the fibers of π are orbits of the action. Moreover assume that the gradient of the projection $\nabla \pi$ is nowhere lightlike. Then, there exists $\tilde{\varepsilon} \in \{\pm 1\}$ and a map $v : I \rightarrow (\mathbb{R}, \tilde{\varepsilon} ds^2)$, such that $v \circ \pi$ is a semi-Riemannian submersion with constant mean curvature fibers.*

Proof. Since $\nabla \pi$ is never zero or lightlike, $\text{sign}(g(\nabla \pi, \nabla \pi)) = \tilde{\varepsilon} = \pm 1$ is constant. In addition, since the fibers of π are orbits of Σ , the length of $\nabla \pi$ is constant along them. Then, let $v : I \rightarrow (\mathbb{R}, \tilde{\varepsilon} ds^2)$ be a function such that $v' = (\sqrt{\tilde{\varepsilon} |\nabla \pi|^2})^{-1}$. It follows that $v \circ \pi$ satisfies $|\nabla(v \circ \pi)|^2 = \tilde{\varepsilon}$. Therefore it is a semi-Riemannian submersion. Next, since the fibers of π are orbits of the Lie group action, the mean curvature must be constant along them. As v' has constant sign, v must be injective, hence the fibers of $v \circ \pi$ are the same as the fibers of π . It follows that its fibers have constant mean curvature $H_{\text{fib}} = \text{div}(\nabla(v \circ \pi))$. \square

From a technical point of view, the proposition shows that there is no loss of generality if we assume that $|\nabla \pi|_g^2 = \tilde{\varepsilon}$. and $\text{div}(\nabla \pi) = h \circ \pi = H_{\text{fib}}^{\nabla \pi}$, for a suitable function $h : I \rightarrow \mathbb{R}$. Note that it is necessary for the Lie group to act by isometries. Indeed, consider the standard flat Riemannian metric on \mathbb{R}^2 and the action of the Boost Group on \mathbb{R}^2 . Then, a simple computation shows that we cannot obtain the constancy of the length of $\nabla \pi$ (see Example 7).

We recall that a function $u : M \rightarrow \mathbb{R}$ is called *invariant* by the Lie group Σ , or also Σ -invariant, if it satisfies

$$u : M \rightarrow \mathbb{R}, \quad u(x) = u(\sigma \cdot x), \quad \forall x \in M, \quad \forall \sigma \in \Sigma. \quad (10)$$

Accordingly, we will say that a graphical translating soliton is *invariant* by the Lie group Σ , or also Σ -invariant, when its graph map is invariant by Σ . We are now ready to prove the following

Theorem 3. *Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : (M, g_M) \rightarrow (I, \tilde{\varepsilon} ds^2)$ be a semi-Riemannian submersion such that the fibers of π are orbits of the action, with function h representing the mean curvature of the orbits. Take $u \in C^2(M, \mathbb{R})$ and consider its graph map $F : M \rightarrow M \times \mathbb{R}$, $F(x) = (x, u(x))$ for any $x \in M$. Then, F is a Σ -invariant translating soliton if, and only if, there exists a solution $f \in C^2(I, \mathbb{R})$ to the ODE (3) such that $u = f \circ \pi$.*

Proof. The fact that u is invariant by Σ readily shows the existence of a smooth function $f : I \rightarrow \mathbb{R}$ such that $u = f \circ \pi = f(\pi)$. By using Proposition 2, we can assume that $\pi : M \rightarrow I$ is a semi-Riemannian submersion with constant mean curvature fibers. Hence, Theorem 3 follows directly from Theorem 2. \square

Proposition 2 actually shows that we can refine Theorem 3 in the following more general, but also more technical way.

Corollary 2. *Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : (M, g_M) \rightarrow (I, \tilde{\varepsilon} ds^2)$ be a submersion such that the fibers of π are orbits of the action, with function $h : I \rightarrow \mathbb{R}$ representing the mean curvature of the fibers. Assume that the gradient of the projection $\nabla \pi$ is nowhere lightlike. Consider $u \in C^2(M, \mathbb{R})$ and its graph map $F : M \rightarrow M \times \mathbb{R}$, $F(x) = (x, u(x))$ for any $x \in M$. Then, F is a Σ -invariant translating soliton if and only if there exists a solution $f \in C^2(I, \mathbb{R})$ to the ODE (3), such that $u = f \circ v \circ \pi$, with $v : I \rightarrow \mathbb{R}$ a map satisfying $v' = (\sqrt{\tilde{\varepsilon} |\nabla \pi|^2})^{-1}$.*

Before studying examples, we want to give some results about the behavior of solutions of the ODE (3) depending on the function h and the signs ε and $\tilde{\varepsilon}$.

Corollary 3. *Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : M \rightarrow I$ be a submersion such that the fibers of π are orbits of the action. Moreover assume that $|\nabla \pi|^2 = \tilde{\varepsilon} = \pm 1$ and $\text{div}(\nabla \pi) = h \circ \pi$, for some function $h : I \rightarrow \mathbb{R}$. Then for each $y_0 \in \mathbb{R}$ and each $s_0 \in I$ such that $h(s_0) \neq 0$, there exist a real number $\rho > 0$ and a translating soliton $F : (y_0 - \rho, y_0 + \rho) \times \Sigma \rightarrow \tilde{M}$ such that it is the union of two graphical translating solitons.*

Proof. If $f'(s_0) \neq 0$, then, there is a small interval J around $y_0 = f(s_0) \in \mathbb{R}$ such that $\alpha = f^{-1}$ is well-defined on it. Given $y \in J$, we have according to (3),

$$-\frac{\alpha''(y)}{\alpha'(y)^3} = \left(\varepsilon + \frac{\tilde{\varepsilon}}{\alpha'(y)^2} \right) \left(1 - \frac{h(\alpha(y))}{\alpha'(y)} \right),$$

which is

$$\alpha''(y) = (\tilde{\varepsilon} + \varepsilon \alpha'(y)^2) (h(\alpha(y)) - \alpha'(y)).$$

From here, we can follow the proof of Lemma 2.3 in [4]. \square

Corollary 4. *Under the same conditions, assume that $\varepsilon \tilde{\varepsilon} = -1$. Then, given $s_0 \in I$, $f_1 \in (-1, 1)$ and $f_0 \in \mathbb{R}$, there exists a solution $f : I \rightarrow \mathbb{R}$ to (3) such that $f(s_0) = f_0$ and $f'(s_0) = f_1$.*

Proof. We make the change $w = f'$, so that (3) reduces to

$$w'(s) = \pm(1 - w(s)^2)(1 - w(s)h(s)).$$

Note that constant functions $w(s) = \pm 1$ are solutions to this differential equation, and they do not cross. Then, given initial conditions $s_0 \in \text{Int}(I)$ and $f_1 \in (-1, 1)$, there exists a well-defined solution $w : \text{Int}(I) \rightarrow \mathbb{R}$. It remains to compute $f(s) = f_0 + \int_{s_0}^s w(x) dx$ for some $f_0 \in \mathbb{R}$. \square

Finally, we are studying finite time blow ups of solutions.

Corollary 5. *Let $\varepsilon = 1$ and $\tilde{\varepsilon} = -1$. Take $c \in I$. Consider any $\lambda > 1$.*

- (1) *Let f be a solution to (3) such that $f'(c) = \lambda$. If $h(s) \leq 0$ for any $s \geq c$, $s \in I$, and $\sup I > c + \coth^{-1}(\lambda) = a$, then f admits a finite time blow up before a .*
- (2) *Let f be a solution to (3) such that $f'(c) = -\lambda$. If $h(s) \leq -1$ for any $s \geq c$, $s \in I$, and $\sup I > c + \coth^{-1}(\lambda)/(\lambda - 1) = a$, then f admits a finite time blow up before a .*

Proof. Again we can write $w = f'$ to reduce to the ODE

$$w'(s) = (-1 + w(s)^2)(1 - w(s)h(s)). \quad (11)$$

(1) First we assume that $w(c) = f'(c) > 1$ for some $c \in I$ and $h(s) \leq 0$ for $s \geq c$, $s \in I$. Take w any solution such of (11) such that $w(c) = \lambda > 1$. Since the constant function 1 is also a solution, then $w(s) > 1$. Now, we have $-w(s)h(s) \leq 0$ for any $s \geq c$, $s \in I$. Hence, by the ODE, $w'(s) \geq 0$. But now, we obtain

$$(-1 + w^2)(1 - hw) \geq -1 + w^2.$$

Therefore the ODE (11) dominates the ODE $g' = -1 + g^2$, the general solution to this equation is

$$g(s) = \frac{1 + e^{2(s-a)}}{1 - e^{2(s-a)}} = \coth(a - s), \quad a \in \mathbb{R}.$$

By imposing $g(c) = w(c)$, we obtain $\coth(a - c) = w(c) = \lambda$, so function g has a finite time blow up at $a = c + \coth^{-1}(\lambda)$.

(2) Take w a solution to (11) such that $w(c) = -\lambda < -1$, for some $c \in I$ and $h(s) \leq -1$ for $s \geq c$. Since the constant function -1 is a solution, it follows that $w(s) < -1$. Then $w'(s) < 0$ by the ODE (11) for all $s \geq c$ and w is decreasing. Hence, we have $w(s) < -\lambda$. Therefore

$$(-1 + w^2)(1 - hw) \leq (-1 + w^2)(1 + h\lambda) \leq -(-1 + w^2)(\lambda - 1).$$

Now, if g is the solution to the problem $g' = (1 - g^2)(\lambda - 1)$, $g(c) = w(c) = -\lambda$, then $-w$ dominates $-g$. Note that the solution to this problem is

$$g(s) = \frac{1 + e^{2(\lambda-1)(s-a)}}{-1 + e^{2(\lambda-1)(s-a)}} = \coth((\lambda - 1)(s - a)), \quad a \in \mathbb{R}.$$

By imposing $g(c) = w(c) = -\lambda$, we obtain that function g has a finite time blow up at $a = c + \frac{\coth^{-1}(\lambda)}{\lambda-1}$. \square

Corollary 6. *Let $\varepsilon = \tilde{\varepsilon} = -1$ and $c \in I$. Consider any $\lambda > 0$ such that $\sup I > c + 1/\lambda = a$.*

- (1) *Let f be a solution to (3) such that $f'(c) = -\lambda$. If $h(s) > 0$ for any $s \geq c$, $s \in I$, then f admits a finite time blow up before a .*
- (2) *Let f be a solution to (3) such that $f'(c) = \lambda$. If $h(s) > 1/f'(c)$ for any $s \geq c$, $s \in I$, then f admits a finite time blow up before a .*

Proof. The proof of case is similar to Corollary 6, but we have to compare with the solutions to the ODE $g' = -g^2$. \square

- Corollary 7.** 1. If $\varepsilon = \tilde{\varepsilon} = 1$, consider any $\lambda > 0$ such that $\sup I > c + 1/\lambda$. Take any solution f to (3). If either $f'(c) = \lambda$ and $h(s) < 0$ for any $s \geq c$, $s \in I$, or $f'(c) = -\lambda$ and $h(s) < -1/f'(c)$ for any $s \geq c$, $s \in I$, then f admits a blow up before a .
2. If $\varepsilon = -1$ and $\tilde{\varepsilon} = 1$, consider any $\lambda > 1$. Let f be a solution to (3) such that $f'(c) = -\lambda$. If $h(s) > 0$ for any $s \geq c$, $s \in I$, and $\sup I > c + \coth^{-1}(\lambda) = a$, then f admits a finite time blow up before a .
3. If $\varepsilon = -1$ and $\tilde{\varepsilon} = 1$, consider any $\lambda > 1$. Let f be a solution to (3) such that $f'(c) = \lambda$. If $h(s) > 1$ for any $s \geq c$, $s \in I$, and $\sup I > c + \coth^{-1}(\lambda)/(\lambda - 1) = a$, then f admits a finite time blow up before a .

Proof. We just need to take $W = -f'$ and $H = -h$, and use the previous corollaries. \square

5 Examples

Example 1. A generalization of the Grim Reaper Cylinder in \mathbb{R}^{m+1} . Given (M, g_M) a connected semi-Riemannian manifold, let Ω be an open domain in M such that $\Gamma : \Omega \rightarrow M \times \mathbb{R}$ is a translating soliton in $(M \times \mathbb{R}, g_M + \varepsilon dt^2)$, for some $\varepsilon = \pm 1$. Now, we consider another connected semi-Riemannian manifold (P, g_P) . The product

$$\bar{\Gamma} : P \times \Omega \rightarrow (P \times M \times \mathbb{R}, g_P + g_M + \varepsilon dt^2) \quad \bar{\Gamma}(q, x) = (q, \Gamma(x)),$$

is just another translating soliton. Indeed, if \vec{H} is the mean curvature vector of Γ , then, $\vec{J} = (0, \vec{H})$ is the mean curvature vector of $\bar{\Gamma}$, since we are dealing with the product spaces. Thus, $\vec{J} = \partial_t^\perp$. \square

Example 2. We study those translating solitons in \mathbb{R}^{n+1} with standard flat metric which are $SO(n)$ -invariant, also known as *rotationally invariant*. If in \mathbb{R}^n with usual coordinates $x = (x_1, \dots, x_n)$, we take $\pi(x) = \sqrt{\sum_i x_i^2}$, then

$$\nabla \pi(x) = \sum_i \frac{x_i}{\pi(x)} \partial_i|_x, \quad |\nabla \pi(x)|^2 = \sum_i \frac{x_i^2}{\pi(x)^2} = 1, \quad \operatorname{div}(\nabla \pi)(x) = \frac{n-1}{\pi(x)}.$$

This means that our manifold has to be $M = \mathbb{R}^n \setminus \{0\}$. However, in order to obtain a C^2 -class translating soliton, (3) becomes

$$f''(s) = \left(1 + f'(s)^2\right) \left(1 - \frac{n-1}{s} f'(s)\right), \quad f'(0) = 0, \quad f(0) = a \in \mathbb{R}. \quad (12)$$

In [2], it was proved that there exist a convex, rotationally symmetric translating soliton over the plane. This was improved in [4] obtaining entire rotationally symmetric translating solitons $F : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^{n+1}$, $n \geq 2$. In fact, in both papers [2] and [4], the same ODE as (12) is obtained. This means that this problem has a unique $C^\infty[0, +\infty)$ solution. The associated translating soliton is known as *translating paraboloid*. In addition, by Corollary 3, we recover the *translating catenoid*.

In the book [3], we can find a list of Lie groups acting transitively and effectively on the sphere \mathbb{S}^{n-1} , hence, also on \mathbb{R}^n , and the space of leaves will be in all cases the interval $[0, +\infty)$. Thus, we can change $O(n)$ by any of them, namely $SO(n)$, $SU(n)$, $Sp(n)Sp(1)$, $Sp(n)U(1)$, $Sp(n)$, G_2 , $Spin(7)$ and $Spin(9)$, for suitable values of the dimension n , depending on the case. This means that we can use all these groups to recover the rotationally symmetric translating solitons in \mathbb{R}^n . \square

In the following examples, we will need some tools, which can be found in the book [11]. In \mathbb{R}^2 , with coordinates $(s, t) \in \mathbb{R}^2$, an *autonomous vector field* is a map $X : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, at least of class C^1 . Given a point $p \in \mathbb{R}^2$ such that $X(p) = (0, 0)$, we compute its *linearlization* at p , namely

$$DX(p) = \left(\frac{\partial X}{\partial s}(p) \quad \frac{\partial X}{\partial t}(p) \right).$$

We can compute the (complex) eigenvalues of this matrix, λ_1 and λ_2 , with eigenvectors v_1 and v_2 . When both eigenvalues are real and $\lambda_1 \lambda_2 < 0$, the point p is called a *saddle point*. In such case, according to Theorem 3.2.1 of [11], there is a submanifold (in this case, just a curve) whose tangent space at p is spanned by the eigenvector of negative eigenvalue, called the *unstable submanifold*. The good property for us is that this submanifold can be seen as the graph of map defined on a small neighbourhood of p .

Example 3. Consider the Minkowski space \mathbb{L}^{n+1} with standard flat metric $g = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2$. A rotationally invariant space-like soliton $F : \mathbb{R}^n \rightarrow \mathbb{L}^{n+1}$, $F(x) = (x, u(x))$ is determined by the group $O(n)$, in a very similar way as in Example 2, but (3) is now

$$f''(s) = (1 - f'(s)^2) \left(1 - \frac{n-1}{s} f'(s) \right), \quad f'(0) = 0, \quad f(0) = a \in \mathbb{R}. \quad (13)$$

To study the solution to this problem, first step is to define $w = f'$, obtaining

$$w'(s) = (1 - w(s)^2) \left(1 - \frac{n-1}{s} w(s) \right), \quad w(0) = 0. \quad (14)$$

We consider the following *autonomous vector field*:

$$X : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad X(s, x) = \left(s, (n-1)x^3 - sx^2 - (n-1)x + s \right).$$

Note that $X(0, 0) = (0, 0)$. At $(0, 0)$, the *linearlization* is

$$DX(0, 0) = \left(\frac{\partial X}{\partial s}(0, 0) \quad \frac{\partial X}{\partial x}(0, 0) \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1-n \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1 - n$, with corresponding eigenvectors $v_1 = (n, 1)^t$, $v_2 = (0, 1)^t$. The point $(0, 0)$ is a saddle point. By Theorem 3.2.1 of [11], there exist a 1-dimensional, local unstable analytic manifold (*of fixed points*), around $(0, 0)$, whose tangent space at $(0, 0)$ is spanned by v_2 , which is a graph in an small interval around $s = 0$, namely

$$W = \{(s, x) \in \mathbb{R} \times \mathbb{R} : x = w(s), |s| \text{ small}\},$$

for some analytic function w defined on a small interval $(-\bar{\delta}, \bar{\delta})$. This means that our dynamical system has a solution $\alpha : (-\delta, \delta) \rightarrow W$, $\alpha(t) = (s(t), x(t))$, with $\alpha'(t) = X(\alpha(t))$, which is an analytic diffeomorphism such that $\alpha(0) = (0, 0)$, $\alpha'(0) = \lambda(n, 1)$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $x(t) = w(s(t))$. We compose with the inverse of s , so that $t = t(s)$. Thus, $w(s) = x(t(s))$ is analytic near 0, with $w(0) = 0$. In addition, since $X(\alpha(t)) = \alpha'(t)$, then for $s > 0$,

$$\begin{aligned} w'(s) &= x'(t(s))t'(s) = \frac{x'(t(s))}{s'(t)} = \frac{(1 - x(t(s))^2)(s(t) - (n-1)x(t(s)))}{s(t)} \\ &= (1 - x(t(s))^2) \left(1 - \frac{(n-1)x(t(s))}{s(t)} \right) = (1 - w(s)^2) \left(1 - \frac{(n-1)w(s)}{s} \right). \end{aligned}$$

From here, recalling $f' = w$, we obtain an analytic solution to (13). Then, as in Corollary 4, we know that we can extend this solution to $f : [0, \infty) \rightarrow \mathbb{R}$, with $f'(0) = 0$.

In a similar fashion to Example 2, we can construct a spacelike rotationally symmetric translating soliton with two ends, that we can also call the *translating catenoid* in \mathbb{L}^{n+1} . \square

Corollary 8. *There do not exist time-like $SO(n)$ -invariant translating solitons in Minkowski space \mathbb{L}^{n+1} .*

Proof. Let $F : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{L}^{n+1}$, $F(x) = (x, u(x))$ a timelike $SO(n)$ -invariant translating soliton. The action of $SO(n)$ in \mathbb{L}^{n+1} is determined by the time-like axis $L = \{(0, x_{n+1}) \in \mathbb{L}^{n+1} : x_{n+1} \in \mathbb{R}\}$. For F be $SO(n)$ -invariant and smooth, then the tangent space at the intersection of $F(\Omega) \cap L$ has to be spacelike, which is a contradiction. \square

Example 4. We consider the Minkowski space \mathbb{L}^{n+1} , $n \geq 3$, with its usual flat metric $g(X, Y) = \sum_{i=1}^n X_i Y_i - X_{n+1} Y_{n+1}$. The de Sitter space-time is

$$dS^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{L}^{n+1} : g(x, x) = +1\}.$$

As usual, we identify the tangent space at $x \in dS^n$,

$$T_x dS^n = \{X = (X_1, \dots, X_{n+1}) \in \mathbb{R}^{n+1} : g(X, x) = 0\}.$$

Now, the Lie group $O(n-1)$ acts by isometries on dS^n as usual:

$$O(n-1) \times dS^n \rightarrow dS^n, \quad (A, x) \mapsto A \cdot x = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) x = \begin{pmatrix} A(x_1, \dots, x_n)^t \\ x_{n+1} \end{pmatrix}.$$

The space of orbits can be identified by the following projection map

$$\pi : dS^n \rightarrow \mathbb{R}, \quad \pi(x) = \sinh^{-1}(x_{n+1}).$$

A straightforward computation shows

$$|\nabla \pi|^2 = -1, \quad \operatorname{div}(\nabla \pi) = -\frac{\sinh(\pi)(n \cosh(\pi)^2 + 1)}{\cosh(\pi)^3}, \quad h(s) = -\frac{\sinh(s)(n \cosh(s)^2 + 1)}{\cosh(s)^3}.$$

Finally, the initial value problem becomes

$$f''(s) = (-1 + \varepsilon f'(s)^2) \left(1 + f'(s) \frac{\sinh(s)(n \cosh(s)^2 + 1)}{\cosh(s)^3} \right), \quad f'(s_o) = f_0, \quad f(s_o) = f_1, \quad (15)$$

for any $s_o \in \mathbb{R}$ and any initial values $f_0, f_1 \in \mathbb{R}$.

We point out that by Corollary 4, when $\varepsilon = +1$, there exist solutions defined on the whole \mathbb{R} whenever $f'(s_o) \in (-1, 1)$. They give rise to entire rotationally symmetric translating solitons in $dS^n \times \mathbb{R}$. Since $\tilde{\varepsilon} = -1$, this type of translating solitons will have $\varepsilon' = \operatorname{sign}(\varepsilon + \tilde{\varepsilon}(f')^2) = +1$, that is to say, all of them will be spacelike. In addition, by Corollary 3, it is possible to construct spacelike rotationally symmetric translating solitons with two ends whose topology is $\mathbb{S}^{n-1} \times \mathbb{R}$.

Note that from Corollary 5 for $\varepsilon = +1$, as well as $\varepsilon = -1$, it is clear that the timelike solutions presents a finite time blow up.

Example 5. In \mathbb{R}^n , $n \geq 2$, with the standard flat metric g_o , we consider the *Poincaré's ball model* of the Hyperbolic Space \mathbb{H}^n , namely

$$\mathbb{D}^n = \{x \in \mathbb{R}^n : g_o(x, x) < 1\}, \quad g = \frac{4}{(1 - |x|^2)^2} g_o.$$

The Lie group $SO(n)$ acts by isometries on \mathbb{D}^n as usual by the standard matrix multiplication:

$$SO(n) \times \mathbb{D}^n \rightarrow \mathbb{D}^n, \quad (A, x) \mapsto Ax.$$

The space of orbits can be identified by the following projection map, $\Omega = \mathbb{D}^n \setminus \{0\}$,

$$\pi : \Omega \rightarrow (0, +\infty), \quad \pi(x) = \ln \left(\frac{1 + |x|}{1 - |x|} \right).$$

A straightforward computation shows

$$|\nabla \pi|^2 = 1 = \tilde{\varepsilon}, \quad \operatorname{div}(\nabla \pi) = \frac{n-1}{4} \coth(\pi), \quad h : (0, +\infty) \rightarrow \mathbb{R}, \quad h(s) = \frac{n-1}{4} \coth(s).$$

Thus, we arrive to the following problem

$$f''(s) = (1 + \varepsilon f'(s)^2) \left(1 - \frac{n-1}{4} \coth(s) f'(s) \right), \quad f'(0) = 0. \quad (16)$$

Next, we want to show that this problem has solutions. Thus, we proceed as in Example 3. We consider the dynamical system

$$X : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad X(s, x) = \left(\sinh(s), (1 + \varepsilon x^2) \left(\sinh(s) - \frac{n-1}{4} \cosh(s)x \right) \right).$$

Note that $X(0, 0) = (0, 0)$. The *linearization* at $(0, 0)$ is

$$DX(0, 0) = \left(\frac{\partial X}{\partial s}(0, 0) \quad \frac{\partial X}{\partial x}(0, 0) \right) = \begin{pmatrix} 1 & 0 \\ 1 & (1-n)/4 \end{pmatrix}.$$

By repeating the steps in Example 3, we obtain an analytic solution $f : [0, T) \rightarrow \mathbb{R}$ to (16) for each $a \in \mathbb{R}$, $f(0) = a$.

When $\varepsilon = -1$, by Corollary 4, $T = +\infty$. This means that we obtain entire rotationally symmetric graphical translating solitons in $\mathbb{D}^n \times \mathbb{R}$ with a Lorentzian metric. In addition, it is possible to construct a *translation helicoid* in this setting.

Assume now that $\varepsilon = +1$. Firstly, we see that $f'(s) \geq 0$ for any $s \in [0, T)$. Indeed, if there exists $s_o \in (0, T)$ such that $f'(s_o) < 0$, by (16), we obtain $f''(s_o) > 0$. Thus, by continuity, there exists $s_1 \in (0, s_o)$ such that $f''(s_1) = 0$ and $f'(s_1) \leq f'(s_o)$. But then, by (16), we obtain $0 = (1 + f'(s_1)^2) \left(1 - \frac{n-1}{4} \coth(s_1) f'(s_1) \right) > 0$, which is a contradiction. Next, we are going to see that we can extend f from 0 to $+\infty$. For the sake of simpleness, we call $w = f' : [0, T) \rightarrow \mathbb{R}$, $g, G : [0, T) \times [0, \infty) \rightarrow \mathbb{R}$ given by $g(s, x) = (1 + x^2)(1 - (n-1) \coth(s)x/4)$ and $G(s, x) = (1 + x^2)(1 - (n-1)x/(4s))$. A simple computation shows

$$G(s, x) - g(s, x) = \frac{n-1}{4} (1 + x^2) \left(\coth(s) - \frac{1}{s} \right) x \geq 0, \quad \forall (s, x) \in [0, \infty) \times [0, \infty).$$

Thus, the solution to the problem $w'(s) = G(s, w(s))$, $w(0) = 0$ is an upper bound of the solution to the problem $w'(s) = g(s, w(s))$, $w(0) = 0$. But we already know that the solution to the problem

$$w(0) = 0, \quad w'(s) = (1 + w(s)^2) \left(1 - \frac{n-1}{4s} w(s) \right)$$

is defined on $[0, \infty)$. This readily shows that f , solution to (16), is defined on the whole $[0, \infty)$. \square

Example 6. We recall

$$H_1^3 = \{(z_1, z_2) \in \mathbb{C}^2 : -|z_1|^2 + |z_2|^2 = -1\}$$

is a 3-dimensional quadric in the standard indefinite complex space \mathbb{C}_1^2 . Also, it is well-known that the unit group of complex numbers $\mathbb{S}^1 = \{a \in \mathbb{C} : a\bar{a} = 1\}$ acts by isometries on H_1^3 , $(a, (z_1, z_2)) \rightarrow (az_1, az_2)$, obtaining a principal fiber bundle over the complex hyperbolic line $H_1^3 \rightarrow \mathbb{C}H^1$ in the usual way, with timelike, totally geodesic fibers \mathbb{S}^1 . In addition, $\mathbb{C}H^1$ is isometric to the real hyperbolic plane $\mathbb{R}H^2$. In fact, by denoting \bar{z} the complex conjugate of $z \in \mathbb{C}$, we can construct the map

$$\pi : H_1^3 \rightarrow \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3, \quad \pi(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 + |z_2|^2).$$

The image of this map is the Weierstrass model of $\mathbb{R}H^2$, embedded in the Minkowski 3-space as a quadric, and therefore it is easy to regard π as the projection from H_1^3 to $\mathbb{C}H^1 \equiv \mathbb{R}H^2$. Next, in Example 5 we obtained rotationally symmetric translating solitons in $\mathbb{R}H^2$, with group $\Sigma = \mathbb{S}^1$. Now, by Corollary 1, we obtain a translating soliton Γ in H_1^3 , by lifting the translating soliton from $\mathbb{R}H^2$. Note that Γ is invariant by the Lie group $\mathbb{S}^1 \times \mathbb{S}^1$. \square

Example 7. Take the Minkowski plane \mathbb{L}^2 with standard flat metric $g = dx^2 - dy^2$. The *boost* Lie group is the set of matrices

$$\Sigma = \left\{ A_\theta = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

which acts on \mathbb{L}^2 by isometries. However, in order to obtain suitable quotients, we need to split the plane in four regions whose boundaries are made of two light-like geodesics, namely

$$\begin{aligned} \Omega_1 &= \{(x, y) \in \mathbb{L}^2 : y^2 < x^2, 0 < x\}, & \Omega_2 &= \{(x, y) \in \mathbb{L}^2 : y^2 > x^2, 0 < y\} \\ \Omega_3 &= \{(x, y) \in \mathbb{L}^2 : y^2 < x^2, 0 > x\}, & \Omega_4 &= \{(x, y) \in \mathbb{L}^2 : y^2 > x^2, 0 > y\}. \end{aligned}$$

We will use the globally defined orthonormal frame $\{\partial_x, \partial_y\}$. For F be Σ -invariant, we have to extend the action of Σ to $(\mathbb{R}^3, \langle, \rangle = g + dt^2) = \mathbb{L}^3$ in the natural way, namely

$$\Sigma \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (A, (x, y, t)) \mapsto ((x, y)A, t).$$

Then, at the intersection point with the axis $\{(0, 0)\} \times \mathbb{R}$, the tangent plane has to be orthogonal to the axis. In other words, $(dF)_{(0,0)} T_{(0,0)} \mathbb{R}^2 \perp \partial_t$. This means that $f'(0) = 0$.

We begin with Ω_1 . We define the projection, with its usual properties:

$$\begin{aligned} \pi : \Omega_1 &\rightarrow (0, +\infty), \quad \pi(x, y) = +\sqrt{x^2 - y^2}, \\ \nabla \pi(x, y) &= \frac{x}{\pi(x, y)} \partial_x + \frac{y}{\pi(x, y)} \partial_y, \quad |\nabla \pi|^2 = +1 = \tilde{\varepsilon}, \quad \operatorname{div}(\nabla \pi) = \frac{1}{\pi}. \end{aligned}$$

It is very simple to check that $\pi((x, y)A_\theta) = \pi(x, y)$ for any $(x, y) \in \mathbb{L}^2$ and $\theta \in \mathbb{R}$. This means that π can work as the expected projection map.

The product manifold $(\Omega_1 \times \mathbb{R}, \langle, \rangle = g + dt^2)$ is an open subset of flat 3-Minkowski space. Thus, we are constructing a time-like translating soliton as a graph over a timelike 2-plane. We transform our differential equation into the following problem.

We recall that we need $W^2 = \varepsilon'(\varepsilon + \tilde{\varepsilon}f'(\pi)^2) \geq 0$. By now, we know $\varepsilon = \tilde{\varepsilon} = +1$. Then, (3) becomes

$$f''(s) = (1 + f'(s)^2) \left(1 - \frac{f'(s)}{s} \right), \quad f(0) = a, \quad f'(0) = 0. \quad (17)$$

We already know that this problem has a $C^\infty[0, +\infty)$ solution by Example 2.

We can construct our translating soliton $F^1 : \Omega_1 \rightarrow \mathbb{L}^3$, $F^1(x, y) = (x, y, f(\sqrt{x^2 - y^2}))$. Note that we have another similar one

$$F^3 : \Omega_3 \rightarrow \mathbb{L}^3, \quad F^3(x, y) = (x, y, f(\sqrt{x^2 - y^2})).$$

In addition, by Corollary 3, there exist *boost invariant, spacelike, translating solitons with two ends*.

Next, we work on Ω_2 . Now, the projection π satisfies

$$\begin{aligned} \pi : \Omega_2 \rightarrow (0, +\infty), \quad \pi(x, y) &= +\sqrt{y^2 - x^2}, \\ \nabla \pi(x, y) &= \frac{-x}{\pi(x, y)} \partial_x - \frac{y}{\pi(x, y)} \partial_y, \quad |\nabla \pi|^2 = -1 = \tilde{\varepsilon}, \quad \operatorname{div}(\nabla \pi) = \frac{-1}{\pi}. \end{aligned}$$

The differential equation (3) becomes now:

$$f''(s) = (-1 + f'(s)^2) \left(1 + \frac{f'(s)}{s} \right), \quad f(0) = a, \quad f'(0) = 0. \quad (18)$$

By the easy change $q(s) = -f(s)$, we transform this problem in

$$q''(s) = (1 - q'(s)^2) \left(1 - \frac{q'(s)}{s} \right), \quad q(0) = -a, \quad q'(0) = 0.$$

By Example 3, we know that this problem has an analytic solution in $[0, +\infty)$.

For $a = 0$, we call f_1 the solution to (17) in $[0, +\infty)$, and f_2 the solution to (18) in $[0, +\infty)$, so that we can define a continuous function

$$u : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad u(x, y) = \begin{cases} f_1(\sqrt{x^2 - y^2}), & (x, y) \in \Omega_1 \cup \Omega_3, \\ 0, & (x, y) \in \partial\Omega_i, i = 1, 2, 3, 4, \\ f_2(\sqrt{y^2 - x^2}), & (x, y) \in \Omega_2 \cup \Omega_4. \end{cases}$$

Note that we get immediately $u \in C^0(\mathbb{R}^2) \cap C^\infty(\cup_{i=1}^4 \Omega_i)$. We want to prove that u is in C^∞ . Let $g(s) := f_1(is)$ in a small neighborhood of 0, where $i = \sqrt{-1}$. Then we have

$$[g(s)]'' = -f_1''(is) = -(1 + f_1'(is)^2)(1 - \frac{f_1'(is)}{is}) = -(1 - g'(s)^2)(1 + \frac{g'(s)}{s})$$

Hence g is a solution to equation (18), and consequently $g = f_2$. But then we get comparing the derivatives of f_1 and f_2 , that $f_1^{(i)}(0) = f_2^{(i)}(0) = 0$, if i is odd, since f_1 and f_2 are real, and $f_1^{(4i+2)}(0) = -f_2^{(4i+2)}(0)$ and $f_1^{(4i)}(0) = f_2^{(4i)}(0)$ for all $i \geq 0$.

Notice that in the same way if $\tilde{f}_1(-s) := f_1(s)$ and $\tilde{f}_2(-s) := f_2(s)$ in a small neighborhood of 0, similar computations show that \tilde{f}_1 and \tilde{f}_2 are also a solution to (17) and (18) respectively. Therefore, $\tilde{f}_1 = f_1$ and $\tilde{f}_2 = f_2$ and f_1 and f_2 are even.

We now just need to prove the following

Lemma 1. *Let f_1, f_2 be functions in $C^{2n}(\mathbb{R})$, such that $f_1^{(k)}(0) = f_2^{(k)}(0) = 0$, if k is odd, $f_1^{(k)}(0) = (-1)^{\frac{k}{2}} f_2^{(k)}(0)$, if k is even. Then the function u defined as above is in $C^n(\mathbb{R}^2)$.*

Proof. We prove the statement by induction over n . The case $n = 0$ is trivially satisfied, since $f_1(0) = f_2(0)$. Moreover let $g_1(z) = \frac{f'_1(z)}{z}$, $g_2(z) = \frac{f'_2(z)}{z}$. We have

$$\begin{aligned}\partial_x f_1(\sqrt{x^2 - y^2}) &= x g_1(\sqrt{x^2 - y^2}), & \partial_x f_2(\sqrt{y^2 - x^2}) &= -x g_2(\sqrt{y^2 - x^2}) \\ \partial_y f_1(\sqrt{x^2 - y^2}) &= -y g_1(\sqrt{x^2 - y^2}), & \partial_y f_2(\sqrt{y^2 - x^2}) &= y g_2(\sqrt{y^2 - x^2})\end{aligned}$$

Obviously g_1 and g_2 are in C^{2n-2} since $f'_1(0) = f'_2(0) = 0$, and $g_1^{(4i+2)}(0) = -g_2^{(4i+2)}(0)$ and $g_1^{(4i)}(0) = g_2^{(4i)}(0)$ for all $i \geq 0$. Then by the induction hypothesis the function

$$v : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad v(x, y) = \begin{cases} g_1(\sqrt{x^2 - y^2}), & (x, y) \in \Omega_1 \cup \Omega_3, \\ 0, & (x, y) \in \partial\Omega_i, i = 1, 2, 3, 4, \\ g_2(\sqrt{y^2 - x^2}), & (x, y) \in \Omega_2 \cup \Omega_4. \end{cases}$$

is in $C^{n-1}(\mathbb{R}^2)$. Hence $\partial_x u$ and $\partial_y u$ extend to $C^{n-1}(\mathbb{R}^2)$, and finally u extends to $C^n(\mathbb{R}^2)$. \square

This proves that u is in C^∞ . As a final remark, we point out that the curve joining each two adjacent pieces is a lightlike straight line. Thus, it is possible to consider two or three contiguous pieces, and their gluing straight lines. We cannot consider the other straight half-lines, since the boundary would be lightlike. In other words, we can choose to glue either two, three or four adjacent pieces to obtain solitons.

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